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Absolutely Proximinal Subspaces of Banach Spaces

J. F. MENA, R. PAYÁ, AND A. RODRÍGUEZ

Departamento de Analisis Matematico, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

AND

D. T. YOST*

Department of Mathematics, I.A.S., Australian National University, GPO Box 4, Canberra 2601, Australia

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Let us say that a subspace M of a Banach space X is absolutely proximinal if it is proximinal and, for each $x \in X$, $\|x\|$ can be expressed as a function of $d(x, M)$, the distance from x to M, and $d(0, P_M(x))$, the distance from the origin to the best approximant set. Then this functional dependence must be given by a suitable norm on $\mathbf{R}^2$. This defines a naturally occurring class of subspaces which includes all L^p-summands, all M-ideals, all subspaces with the $1\frac{1}{2}$-ball property, and all absolute subspaces. This paper initiates the study of this class of subspaces. Amongst other things, we show that:

- The set-valued metric projection onto an absolutely proximinal subspace is Lipschitz continuous in the Hausdorff metric;
- Absolutely proximinal subspaces are, modulo renorming, the same as subspaces with the $1\frac{1}{2}$-ball property;
- A subspace is absolutely proximinal if and only if its polar is absolutely proximinal in the dual space.

We also obtain some numerical estimates for the inner radius of a set of best approximants. © 1991 Academic Press, Inc.

0. INTRODUCTION

In what follows, $(X, \|\cdot\|)$ denotes a Banach space over the field $\mathbf{K}$ ($\mathbf{R}$ or $\mathbf{C}$). Given a closed subspace M of X, the set of best approximants in M to a vector $x \in X$ is denoted by $P_M(x)$, that is,

$$P_M(x) = \{m \in M : \|x - m\| = \|x + M\|\}.$$

* Present address: FB Mathematik I, Freie Universität Berlin, Arnimallee 3, 1000 Berlin 33, Germany.

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Recall that M is said to be *proximinal* if $P_M(x)$ is nonempty for all $x \in X$. One has easily that $P_M(x)$ is a convex, closed, and bounded set and that

$$P_M(\lambda x + m) = \lambda P_M(x) + m \qquad \text{for all} \quad x \in X, m \in M, \text{and } \lambda \in \mathbf{K}.$$

For any subspace M of X, we define its metric complement by

$$M^{\perp} = \{x \in X : \|x\| = d(x, M)\} = \{x : 0 \in P_M(x)\}.$$

Clearly M is proximinal if and only if $M + M^{\perp} = X$. We always have $M \cap M^{\perp} = \{0\}$. In general, $M^{\perp}$ is not convex, let alone a subspace.

We next consider the function

$$d(0, P_M(x)) = \inf\{\|m\| : m \in P_M(x)\} \qquad (x \in X).$$

We are interested in those proximinal subspaces M with the property that the norm of each vector $x \in X$ depends only on the distances $d(0, P_M(x))$ and $\|x + M\|$. More concretely, a proximinal subspace M of X will be called *absolutely proximinal* if there is a real valued function $f(r, s)$ defined for $r, s \geqslant 0$ such that

$$\|x\| = f(d(0, P_M(x)), d(x, M)) \qquad \text{for all} \quad x \in X. \qquad ♣$$

If necessary we emphasize the function f by saying that M is *f-proximinal*. Our first task, in Section 1, is to find a characterization of those functions f which can appear in ♣. We ignore the trivial cases $M = \{0\}$ and $M = X$. It turns out that f must correspond to a lattice norm on $\mathbf{R}^2$—with, of course, $f(0, 1) = f(1, 0) = 1$.

The special case when f is the L-norm on $\mathbf{R}^2$ (i.e., $L(a, b) = |a| + |b|$) has already been studied. It was shown in [11, Corollary 4] that L-proximinality is equivalent to the $1\frac{1}{2}$-ball property, which was first defined in [23].

The $1\frac{1}{2}$-ball property is in turn a generalization of the notions of the M-ideal and the L-summand, which have been studied by a number of authors [2, 16]. Let us recall that, given a norm $|\cdot|$ on $\mathbf{R}^2$, a $|\cdot|$-summand in X is a complemented subspace M with projection P which satisfies $\|x\| = |(\|Px\|, \|x - Px\|)|$ for all $x \in X$. A $|\cdot|$-ideal is a subspace M whose polar M^0 is a $|\cdot|^*$-summand in X^*. Here $|\cdot|^*$ is the dual norm of $|\cdot|$, defined by $|(r, s)|^* = \max\{|sa + rb| : |(a, b)| = 1\}$. For the M-norm on $\mathbf{R}^2$ (i.e., $M(a, b) = \max\{|a|, |b|\}$), we obtain M-summands and M-ideals. An M-ideal is said to be proper if it is not an M-summand: a typical example is $c_0 \subset l_\infty$. For the L-norm on $\mathbf{R}^2$, it turns out that every L-ideal is already an L-summand. (See, for example, [16, Theorem 6.16].)

A comprehensive study of $|\cdot|$-summands, $|\cdot|$-ideals, and their natural generalizations was undertaken in [19] and [20]. The most general subspaces considered in [20] are the so-called absolute subspaces, and it was proved there that they are absolutely proximinal. Absolute subspaces are not considered until quite late in this paper, so we postpone their definition for the time being. Instead, let us just summarize the relationships which exist between these classes of subspaces:

(i) Every M-ideal is an absolute subspace and has the $1\frac{1}{2}$-ball property.

(ii) Every L-summand is an absolute subspace and has the $1\frac{1}{2}$-ball property.

(iii) Every absolute subspace is absolutely proximinal.

(iv) Every subspace with the $1\frac{1}{2}$-ball property is absolutely proximinal.

It has long been known that M-ideals are proximinal and that the best approximation mapping P_M is well-behaved in a certain sense [14, 17]. (Since L-summands are *Chebyshev*, i.e., $P(x)$ is always a singleton, their approximation theoretic behaviour is less interesting.) Later [23, Sect. 1] it was shown that the good approximation theoretic behaviour of M-ideals is shared by subspaces with only the $1\frac{1}{2}$-ball property.

In this paper, we show that these properties are also shared by absolutely proximinal subspaces. In Section 2 we establish the basic properties of absolutely proximinal subspaces, including the fact that their best approximation operator is Lipschitz continuous, and other results stated in the abstract.

In Section 3 we show that absolutely proximinal subspaces of complex Banach spaces are far more numerous than previously thought. More precisely, we show that every complex Banach space has the $1\frac{1}{2}$-ball property (without being an M-ideal or an L-summand) in some superspace.

In Section 4 we introduce a related but very weak property which we find useful for studying the existence of interior points in $P_M(x)$. Specifically, we obtain some estimates for the inner radius of the set of best approximants. We also show that a Banach space which is absolutely proximinal in its second dual must already have the $1\frac{1}{2}$-ball property in its second dual.

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1. f-Proximinal, $|\cdot|$-Proximinal, and U-Proximinal Subspaces

Here we determine which functions f can appear in the definition of f-proximinality and show that absolutely proximinal subspaces form a subclass of the previously studied U-proximinal subspaces.

LEMMA 1.1. *Let M be a nontrivial proximinal subspace of X, and let r_1, r_2, s, t be nonnegative real numbers with $r_1 < r_2$ and $s \leqslant t$. Then there are x_1, x_2, y in X such that*

$$\text{(i)} \qquad \|y\| = t, \qquad \|y + M\| = s,$$

$$\text{(ii)} \qquad \|x_1 + M\| = \|x_2 + M\| = s, \qquad d(0, P_M(x_1)) = r_1,$$

$$d(0, P_M(x_2)) = r_2, \qquad \|x_1\| < \|x_2\|.$$

Proof. Let $x_0 \in X$ be such that $\|x_0 + M\| = s$, choose $m_0 \in P_M(x_0)$, and write $x = x_0 - m_0$. Also fix $m \in M$ with $\|m\| = 1$ and define

$$\varphi(\lambda) = d(\lambda m, P_M(x)), \ \psi(\lambda) = \|x - \lambda m\| - s, \qquad \text{for all} \quad \lambda \geqslant 0.$$

It can be easily verified that φ, ψ are nonnegative, continuous, unbounded convex functions satisfying $\varphi(0) = \psi(0) = 0$. Moreover $\varphi(\lambda) = 0$ if and only if $\lambda m \in P_M(x)$, if and only if $\psi(\lambda) = 0$. So if we write $\lambda_0 = \max\{\lambda \geqslant 0 : \varphi(\lambda) = 0\}$ $= \max\{\lambda \geqslant 0 : \psi(\lambda) = 0\}$, then φ and ψ are strictly increasing functions for $\lambda \geqslant \lambda_0$. Thus we can find $\lambda_1, \lambda_2, \mu \geqslant \lambda_0$ satisfying

$$\psi(\mu) = t - s, \qquad \varphi(\lambda_1) = r_1, \qquad \varphi(\lambda_2) = r_2, \qquad \lambda_1 < \lambda_2.$$

Finally we take $y = x - \mu m$, $x_1 = x - \lambda_1 m$, $x_2 = x - \lambda_2 m$. ∎

LEMMA 1.2. *Let M be an absolutely proximinal subspace of X. Then there is a unique function f such that M is f-proximinal. Moreover f is strictly increasing and continuous in the first variable.*

Proof. For $r, s \geqslant 0$ we can use Lemma 1.1 to find an $x \in X$ such that $d(0, P_M(x)) = r$ and $\|x + M\| = s$. Then $f(r, s)$ is uniquely determined by the equation

$$f(r, s) = f(d(0, P_M(x)), \|x + M\|) = \|x\|.$$

In the notation of Lemma 1.1 we have

$$f(r_1, s) = \|x_1\| < \|x_2\| = f(r_2, s),$$

so $r \mapsto f(r, s)$ is a strictly increasing function. In view of the first part of the same lemma, the range of this function is an interval, i.e., has no discontinuities. ∎

It is clear that M is an f-proximinal subspace of X if and only if M is an f-proximinal subspace of $M + \mathbf{K}x$ for all x in X. In particular, f-proximinality of M is preserved when we replace X by a closed subspace of X containing M. Also, f-proximinality is obviously preserved when we pass to the real restriction of a complex space. Our next goal is to prove that f-proximinality is also preserved under the formation of quotient spaces.

LEMMA 1.3. *Let M be an absolutely proximinal subspace of X and N a closed subspace of M. Let $Q: X \to X/N$ denote the quotient mapping. Then, for any x in X, $P_{Q(M)}(Q(x))$ is the closure (in X/N) of $Q(P_M(x))$.*

Proof. We have easily $\|Q(x)+Q(M)\| = \|x+M\|$, so $Q(P_M(x)) \subseteq P_{Q(M)}(Q(x))$ for all $x \in X$. Since $P_{Q(M)}(Q(x))$ is closed, it only remains to prove that every element of this set is in the closure of $Q(P_M(x))$. After translation we can suppose that the given element is zero. So we assume that $0 \in P_{Q(M)}(Q(x))$, that is, $\|x+M\| = \|x+N\|$, and we must find elements in N arbitrarily close to $P_M(x)$. Let (n_k) be a sequence in N such that $\|x-n_k\| \to \|x+M\|$. Then $f(d(0, P_M(x-n_k)), \|x+M\|) \to \|x+M\|$, where f is the function given by Lemma 1.2. Choose m in $P_M(x)$ and note that

$$\|x+M\| = \|x-m\| = f(d(0, P_M(x-m)), \|x+M\|) = f(0, \|x+M\|).$$

According to Lemma 1.2, the function $r \mapsto f(r, \|x+M\|)$ (defined for $r \geqslant 0$) has a continuous inverse. So we obtain $d(0, P_M(x-n_k)) \to 0$, that is, $d(n_k, P_M(x)) \to 0$, as required. ∎

Remark 1.4. The assertion of the above lemma is no longer true if we assume M to be only a proximinal subspace of X, as the following example shows. Let $X = l_1$,

$$M = \left\{x \in X : x_1 + \sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) x_n = 0\right\},$$

$$N = \left\{x \in M : \sum_{n=2}^{\infty} x_n = 0\right\},$$

and let $x \in l_1$ be given by $2x_1 = x_2 = 1$ and $x_n = 0$ for $n \geqslant 3$. It is not difficult to verify that $\|x+M\| = \|x+N\| = 1$, and so $0 \in P_{Q(M)}(Q(x))$, whereas $d(0, Q(P_M(x))) = 1$. Thus the best approximation mapping onto an *absolutely* proximinal subspace behaves particularly well under quotients. Note also that the assertion of the above lemma is clearly true when M and N are both proximinal subspaces. The point is that N need not be proximinal in Lemma 1.3.

PROPOSITION 1.5. *Let M be an f-proximinal subspace of X and N a closed subspace of M. Then M/N is an f-proximinal subspace of X/N.*

Proof. Let Q denote again the quotient mapping from X onto X/N. For $x \in X$ we have

$$\begin{aligned} \|Q(x)\| &= \inf_{n \in N} f(d(0, P_M(x+n)), \|x+M\|) \\ &= f(\inf\{d(0, P_M(x+n)) : n \in N\}, \|x+M\|) \\ &= f(\inf\{\|m+n\| : m \in P_M(x), n \in N\}, \|x+M\|) \\ &= f(d(0, Q(P_M(x))), \|x+M\|), \end{aligned}$$

where we have used Lemma 1.2 for the second equality and the rest is obvious. We have already noted that $\|x+M\| = \|Q(x)+Q(M)\|$. An application of Lemma 1.3 then yields $d(0, Q(P_M(x))) = d(0, P_{Q(M)}(Q(x)))$. Thus we have, for all $x \in X$,

$$\|Q(x)\| = f(d(0, P_{Q(M)}(Q(x))), \|Q(x)+Q(M)\|),$$

as required. ∎

The way is now prepared for the determination of those functions f for which there is a nontrivial f-proximinal subspace. By *absolute norm* we mean a norm $(r, s) \mapsto |(r, s)|$ on $\mathbf{R}^2$ satisfying

$$|(r, s)| = |(|r|, |s|)| \quad (\forall r, s \in \mathbf{R}) \qquad \text{and} \qquad |(1, 0)| = |(0, 1)| = 1.$$

PROPOSITION 1.6. *Let f be a real valued function defined on the positive quadrant of $\mathbf{R}^2$. Then the following statements are equivalent.*

(i) *There is a Banach space X with a nontrivial f-proximinal subspace M.*

(ii) *f is the restriction to the positive quadrant of some absolute norm $|\cdot|$, for which $(0, 1)$ is an extreme point of the unit ball of $(\mathbf{R}^2, |\cdot|)$.*

Proof. (i) ⇒ (ii) Without loss of generality, we may suppose that the scalars are real. Passing to a quotient space, Proposition 1.5 allows us to assume that M is one-dimensional. Choose $x \in M$ with $\|x\| = 1$ and $y \in X$ with $d(y, M) = 1$. Then $P_M(y)$ must be an interval of the form $[\lambda, \mu]\, x = \{\alpha x : \lambda \leqslant \alpha \leqslant \mu\}$. Translating y parallel to M, we may suppose that this interval is symmetric about the origin; i.e., that $-\lambda = \mu = n^*$ for some $n^* \geqslant 0$. For any $(\alpha, \beta) \in \mathbf{R}^2$, we have

$$d(\alpha x + \beta y, M) = |\beta|$$

and

$$\begin{aligned} d(0, P_M(\alpha x + \beta y)) &= d(-\alpha x, [-|\beta|\, n^*, |\beta|\, n^*]\, x) \\ &= (|\alpha| - |\beta|\, n^*)^+. \end{aligned}$$

Thus $\|\alpha x+\beta y\| = f((|\alpha| - |\beta|\, n^*)^+, |\beta|)$ depends only on $|\alpha|$ and $|\beta|$. Hence the formula $|(\alpha, \beta)| = \|(|\alpha| + n^*|\beta|)\,x + \beta y\|$ defines an absolute norm on $\mathbf{R}^2$. (Note that $|(0, 1)| = 1$ because $-n^*x \in P_M(y)$.) Clearly $f(\alpha, \beta) = |(\alpha, \beta)|$ for all (α, β) in the positive quadrant.

Finally, we note that, for all $z \in X$, $\|z\| = d(z, M) \Leftrightarrow d(0, P_M(z)) = 0$. Thus $|(\alpha, \beta)| = |\beta| \Leftrightarrow \alpha = 0$, i.e., $(0, 1)$ is an extreme point of $(\mathbf{R}^2, |\cdot|)$.

(ii) $\Rightarrow$ (i) Just take $X = \mathbf{R}^2$ with the norm $|\cdot|$, and $M = \mathbf{R} \oplus \{0\}$. ∎

It is pertinent to observe that absolutely proximinal subspaces form a subclass of the U-proximinal subspaces studied by Lau [15]. Recall that a subspace M of X is said to be U-proximinal if there is a function $\varepsilon: \mathbf{R}^+ \to \mathbf{R}^+$, with $\varepsilon(\rho) \to 0$ as $\rho \to 0$, such that $(1+\rho)\,B \cap (B+M) \subseteq B + \varepsilon(\rho)(B \cap M)$, where $B = B(0, 1)$ denotes the unit ball of X. This property was later rediscovered in [9, Sect. 4].

LEMMA 1.7. *Let $|\cdot|$ be any absolute norm on $\mathbf{R}^2$, for which $(0, 1)$ is an extreme point of the unit ball. Then*

$$\varepsilon(\rho) = \max\left\{\frac{\alpha\rho}{1+\rho-\beta} : |(\alpha, \beta)| \leqslant 1+\rho,\ \beta \leqslant 1\right\} \to 0 \qquad as \quad \rho \to 0.$$

Proof. Let $\alpha(\rho) = \max\{\alpha : |(\alpha, 1-\sqrt{\rho})| \leqslant 1+\rho\}$. Then $\alpha(\rho) \to 0$ as $\rho \to 0$; for otherwise we could find $\alpha \neq 0$ with $|(\alpha, 1-\sqrt{\rho})| \leqslant 1+\rho$ for all sufficiently small ρ. But then $|(\alpha, 1)| \leqslant 1$, contrary to hypothesis.

It follows that $\alpha_1(\rho) = \max\{\alpha(\rho), \rho(1+\rho)/(\rho+\sqrt{\rho})\} \to 0$ as $\rho \to 0$. It suffices now to show that $\varepsilon(\rho) \leqslant \alpha_1(\rho)$.

Given (α, β) as specified above, we consider two cases. First, suppose that $\beta \leqslant 1-\sqrt{\rho}$. Then $\alpha\rho/(1+\rho-\beta) \leqslant \rho(1+\rho)/(\rho+\sqrt{\rho}) \leqslant \alpha_1(\rho)$, as required. In the second case, $\beta \geqslant 1-\sqrt{\rho}$, we have

$$|(\alpha, 1-\sqrt{\rho})| \leqslant |(\alpha, \beta)| \leqslant 1+\rho,$$

and so $\alpha \leqslant \alpha(\rho)$. Then

$$\frac{\alpha\rho}{1+\rho-\beta} \leqslant \frac{\alpha(\rho)\,\rho}{1+\rho-1} \leqslant \alpha_1(\rho),$$

and the proof is complete. ∎

PROPOSITION 1.8. *Every absolutely proximinal subspace is U-proximinal.*

Proof. Let M be $|\cdot|$-proximinal in X, and define $\varepsilon(\rho)$ as in Lemma 1.7. Given x in $(1+\rho)\,B \cap (B+M)$, let us put $\alpha = d(0, P_M(x))$ and $\beta = d(x, M)$.

Then $|(\alpha, \beta)| \leqslant 1+\rho$ and $\beta \leqslant 1$. Given $\delta > 0$, we can find $m \in P_M(x)$ such that $\|m\| < \alpha + \delta$. Put $\tilde{m} = \lambda m$, where $\lambda = \rho/(1+\rho-\beta) \in [0, 1]$. Then

$$\begin{aligned}\|\tilde{m} - x\| &= \|\lambda(m-x) - (1-\lambda)\,x\| \\ &\leqslant \lambda\beta + (1-\lambda)\,|(\alpha, \beta)| \\ &\leqslant \lambda\beta + (1-\lambda)(1+\rho) \\ &= 1,\end{aligned}$$

and $\|\tilde{m}\| = \lambda\,\|m\| < \rho(\alpha+\delta)/(1+\rho-\beta) \leqslant \varepsilon(\rho) + \delta$. Hence $x = x - \tilde{m} + \tilde{m} \in B + (\varepsilon(\rho)+\delta)(B \cap M)$. Choosing δ sensibly as a function of ρ, we see that M is U-proximinal. ∎

Let $H(X)$ denote the family of all bounded, closed, convex subsets of the Banach space X. A metric d can be defined on $H(X)$ by

$$d(A, B) = \sup(\{d(a, B) : a \in A\} \cup \{d(b, A) : b \in B\}) \qquad (A, B \in H(X)).$$

Lau [15] showed that every U-proximinal subspace is actually proximinal and that the metric projection $P\colon X \to H(M)$ is continuous (and so, by [21], admits a continuous selection). The same is therefore true for absolutely proximinal subspaces. Later we give a direct proof of a stronger result: namely, the metric projection onto an absolutely proximinal subspace is Lipschitz continuous. This was already known for subspaces with the $1\frac{1}{2}$-ball property [23]. Combining this with some results from [15] and [18], we see that not every U-proximinal subspace is absolutely proximinal.

2. Principal Properties of Absolutely Proximinal Subspaces

Let us recall the following concepts, from [1] and [19], which are useful in our discussion of absolutely proximinal subspaces. The numerical range ideas underlying the following definitions can be found, for example, in [4] and [5].

Let u be a fixed norm-one element in the Banach space X. We denote by $D(X, u)$ (or simply $D(u)$) the state space of u; that is,

$$D(u) = \{f \in X^* : \|f\| = f(u) = 1\}.$$

Then $D(u)$ is a nonempty, convex, and w^*-compact subset of X^*. For $x \in X$ we write

$$V(u, x) = \{f(x) : f \in D(u)\},$$

which is a compact convex subset of **K**. One could refer to $V(u, x)$ as the numerical range of x with respect to u. We also write

$$M^u(x) = \max\{\text{re }\lambda : \lambda \in V(u, x)\};$$

it is well known [8, Chap. V] that

$$M^u(x) = \lim_{\alpha \downarrow 0} \frac{1}{\alpha} (\|u + \alpha x\| - 1).$$

Finally, if M is a nonzero subspace of X we define a seminorm ρ_M on X by

$$\rho_M(x) = \sup \{M^u(x) : u \in M, \|u\| = 1\}.$$

Since $V(\lambda u, x) = \bar{\lambda} V(u, x)$ whenever $|\lambda| = 1$, it is easily verified that ρ_M is a seminorm. Note that $\rho_M(m) = \|m\|$ for all $m \in M$ and that $\rho_M(x) \leqslant \|x\|$ on X.

Given an absolute norm $|\cdot|$ on $\mathbf{R}^2$, we define two indices $n = n(|\cdot|)$ and $n^* = n^*(|\cdot|)$ as follows:

$$n = \lim_{\alpha \downarrow 0} \frac{|(1, \alpha)| - 1}{\alpha}$$

$$n^* = \max\{\alpha : |(\alpha, 1)| = 1\}.$$

It turns out that $n^*(|\cdot|) = n(|\cdot|^*)$.

We say that $|\cdot|$ is of type 1 if $(1, 0)$ is not a smooth point of the unit ball of $(\mathbf{R}^2, |\cdot|)$, of type 2 if $(1, 0)$ is both an extreme point and a smooth point of the unit ball, and of type ∞ if $(1, 0)$ is not an extreme point of the unit ball. Similarly we define the cotype of $|\cdot|$ according to the behaviour of $(0, 1)$. The analogy with the L^1, L^2, and L^∞ unit balls should be clear. Note that $n > 0$ iff $|\cdot|$ has type 1, $n^* > 0$ iff $|\cdot|$ has cotype ∞, and $n + n^* \leqslant 1$ always. Proposition 1.6 asserts that a nontrivial $|\cdot|$-proximinal subspace exists if and only if the cotype of $|\cdot|$ is not ∞.

The following lemma follows from the above definitions via some calculations with norm derivatives.

LEMMA 2.1. *Let M be a $|\cdot|$-proximinal subspace of X and $u \in M$ with $\|u\| = 1$. Then*

$$M^u(x) = \inf\{M^u(y) : y \in P_M(x)\} + n \|x + M\|$$

for all $x \in X$.

Proof. Let us fix $x \in X$. For $y \in P_M(x)$, the function

$$G_y(\alpha) = |(\|u + \alpha y\|, \alpha d(x, M))| \qquad (\alpha > 0)$$

is convex, so we have

$$\inf\left\{\frac{1}{\alpha}(G_y(\alpha) - 1) : \alpha > 0\right\} = \lim_{\alpha \downarrow 0} \frac{G_y(\alpha) - 1}{\alpha}.$$

By [19, Lemmas 1.6 and 1.5] the limit on the right-hand side equals

$$M^u(y) + nd(x, M).$$

Now, from the definition of $|\cdot|$-proximinality we have

$$\begin{aligned} M^u(x) &= \inf\left\{\frac{1}{\alpha}(\|u + \alpha x\| - 1) : \alpha > 0\right\} \\ &= \inf\left\{\inf\left\{\frac{|(\|u + \alpha y\|, \alpha d(x, M))| - 1}{\alpha} : y \in P_M(x)\right\} : \alpha > 0\right\} \\ &= \inf\left\{\inf\left\{\frac{G_y(\alpha) - 1}{\alpha} : \alpha > 0\right\} : y \in P_M(x)\right\} \\ &= \inf\{M^u(y) + nd(x, M) : y \in P_M(x)\}, \end{aligned}$$

as required. ∎

The next lemma follows from a routine application of the Hahn–Banach and Bishop–Phelps Theorems.

LEMMA 2.2. *Let M be a proximinal subspace of X and let $x \in X$, $\varepsilon > 0$ be such that $\varepsilon < d(0, P_M(x))$. Then there are elements u in the unit sphere of M and $g \in D(M, u)$ such that* $\operatorname{re} g(y) \geqslant \varepsilon$ *for all $y \in P_M(x)$.*

Proof. Let δ be such that $\varepsilon < \delta < d(0, P_M(x))$. Then the open ball in M centred at the origin with radius δ does not meet the set $P_M(x)$, so we can use the Hahn–Banach Separation Theorem to find an $f \in M^*$ with $\|f\| = 1$ such that

$$\begin{aligned} \delta &= \sup\{\operatorname{re} f(m) : m \in M, \|m\| < \delta\} \\ &\leqslant \inf\{\operatorname{re} f(y) : y \in P_M(x)\}. \end{aligned}$$

Using the Bishop–Phelps Theorem [5, Sect. 16] we obtain a u in the unit sphere of M and $g \in D(M, u)$ such that $\|g - f\| < (\delta - \varepsilon)/k$, where $k = \sup\{\|y\| : y \in P_M(x)\}$. Then, for all y in $P_M(x)$ we have

$$\operatorname{re} g(y) \geqslant \delta - \|g - f\| \, \|y\| > \varepsilon,$$

as required. ∎

The following is our fundamental result about absolutely proximinal subspaces. Its technical nature is forgiven in view of its consequences.

THEOREM 2.3. *Let M be a $|\cdot|$-proximinal subspace of X. Then*

$$\max\{\rho_M(x), n\,\|x+M\|\} = d(0, P_M(x)) + n\,\|x+M\|$$

for all $x \in X$. Equivalently, $d(0, P_M(x)) = (\rho_M(x) - nd(x, M))^+$.

Proof. First suppose that $0 \notin P_M(x)$, and choose ε with $0 < \varepsilon < d(0, P_M(x))$. Let u, g be given by Lemma 2.2. By Lemma 2.1 we have $M^u(x) \geq \varepsilon + n\,\|x+M\|$, whence $\rho_M(x) \geq \varepsilon + n\,\|x+M\|$. Now letting $\varepsilon \to d(0, P_M(x))$, we obtain

$$\max\{\rho_M(x), n\,\|x+M\|\} \geq d(0, P_M(x)) + n\,\|x+M\|.$$

This inequality is clear when $0 \in P_M(x)$.

For the reverse inequality, let us fix a norm-one element u in M. Using the fact that $M^u(y) \leq \|y\|$ for all $y \in P_M(x)$, Lemma 2.1 yields $M^u(x) \leq \|y\| + n\,\|x+M\|$ for all $y \in P_M(x)$, whence

$$M^u(x) \leq d(0, P_M(x)) + n\,\|x+M\|.$$

The rest is clear. ∎

THEOREM 2.4. *Let M be a $|\cdot|$-proximinal subspace of X. Then*

$$d(P_M(x), P_M(y)) \leq (1+n)\,\|x-y\|$$

for all $x, y \in X$.

Proof. Let $x, y \in X$ be given, and choose $a \in P_M(x)$. It clearly suffices to show that $d(a, P_M(y)) \leq (1+n)\,\|x-y\|$. If $y - a \in M^\perp$ then $a \in P_M(y)$ and there is nothing to prove. If $y - a \notin M^\perp$ then, using Theorem 2.3 twice, we have

$$\begin{aligned} d(a, P_M(y)) + nd(y, M) &= \rho_M(y-a) \\ &\leq \rho_M(y-x) + \rho_M(x-a) \\ &\leq \|x-y\| + nd(x-a, M) \\ &= \|x-y\| + nd(x, M) \\ &\leq (1+n)\,\|x-y\| + nd(y, M). \end{aligned}$$

Thus $d(a, P_M(y)) \leq (1+n)\,\|x-y\|$, as required. ∎

COROLLARY 2.5. *If the absolute norm* $|\cdot|$ *is not of type* 1, *then every* $|\cdot|$*-proximinal subspace is a Chebyshev* $|\cdot|$*-summand. In particular, if* M *is a proximinal subspace of a Banach space* X *satisfying*

$$\|x\|^p = d(0, P_M(x))^p + d(x, M)^p$$

for all $x \in X$ *and some* p *with* $1 < p < \infty$, *then* M *is an* L^p*-summand of* X.

Proof. We must have $n = 0$ for norms which are not of type 1. Thus Theorem 2.3 becomes $\rho_M(x) = d(0, P_M(x))$ for all $x \in X$. Then $M^\perp = \{x \in X : \rho_M(x) = 0\}$ is a subspace of X, so $X = M \oplus M^\perp$. This shows that M is a Chebyshev subspace, and its (single-valued) metric projection must be a $|\cdot|$-projection. ∎

The second part of Corollary 2.5 improves [19, Corollary 1.9], where it was already assumed that M was a Chebyshev subspace of X.

COROLLARY 2.6. *Let* $|\cdot|$ *be a type* 1 *absolute norm and* M *a* $|\cdot|$*-proximinal subspace of* X. *Define a new norm on* X *by*

$$|||x||| = \max\{\rho_M(x), n\,\|x + M\|\}.$$

Then $|||\cdot|||$ *is an equivalent norm on* X *and* M *has the* $1\frac{1}{2}$*-ball property in* $(X, |||\cdot|||)$. *Moreover* $|||m||| = \|m\|$ *for all* $m \in M$.

Proof. Let $d'(x, M)$ and $P'_M(x)$ denote the new distances and best approximant sets under $|||\cdot|||$. Clearly

$$d'(x, M) = \inf_m (d(m, P_M(x)) + nd(x, M)) = nd(x, M),$$

and so

$$m \in P'_M(x) \Leftrightarrow d(m, P_M(x)) + nd(x, M) = d'(x, M)$$
$$\Leftrightarrow m \in P_M(x).$$

Thus $|||x||| = d(0, P'_M(x)) + d'(x, M)$, as required. ∎

By Corollary 2.5, if the absolute norm is not of type 1, then every $|\cdot|$-proximinal subspace is complemented. It is clear that every complemented subspace satisfies the $1\frac{1}{2}$-ball property under renorming. So we have

PROPOSITION 2.7. *Let* M *be an absolutely proximinal subspace of* X. *Then* X *can be equivalently renormed so as to have* M *satisfy the* $1\frac{1}{2}$*-ball property.*

A renorming process in the direction opposite to that of Proposition 2.7 is restricted to type 1 absolute norms, in view of Corollary 2.5. Under this restriction, we show below that such a renorming process is always possible. It was proved in [19, Lemma 1.10] that, given an absolute norm $|\cdot|$, there is a unique absolute norm $|\cdot|'$ which satisfies

$$|(r, s)| = |(r + ns, s)|^{+}$$

for all $r, s \geqslant 0$. It is easy to see that there is a unique absolute norm $|\cdot|$ for which

$$|(r + n^{*}s, s)| = |(r, s)|$$

whenever $r, s \geqslant 0$. We note that the unit ball of $(\mathbf{R}^2, |\cdot|^{+})$ is closer to the unit ball of $(\mathbf{R}^2, M)$, whereas the unit ball of $(\mathbf{R}^2, |\cdot|^{-})$ is closer to the unit ball of $(\mathbf{R}^2, L)$.

PROPOSITION 2.8. *Let M be a closed subspace of X satisfying the $1\frac{1}{2}$-ball property and let $|\cdot|$ be an absolute norm of type* 1 *and not of cotype* ∞. *Define $|||\cdot|||$ on X by*

$$|||x||| = \left|\left(\|x\|, \frac{1}{n} d(x, M)\right)\right|^{+}.$$

Then $|||\cdot|||$ is an equivalent norm on X and M is a $|\cdot|$-proximinal subspace of $(X, |||\cdot|||)$.

The proof of this is quite similar to that of Corollary 2.6.

Propositions 2.7 and 2.8 show that the class of absolutely proximinal subspaces is essentially the same as the class of subspaces with the $1\frac{1}{2}$-ball property. We note here that the class of U-proximinal subspaces is strictly larger. For example, let X be a uniformly convex space and M an uncomplemented subspace. Then M is U-proximinal [15, Proposition 4.3] and Chebyshev in X. According to [18, Corollary 2], its metric projection cannot be Lipschitz continuous. Theorem 2.4 then shows that M is not absolutely proximinal. This argument remains valid under any renorming of X which preserves the norm on M and the (singleton) sets of best approximants.

By taking into account the existence of uncomplemented subspaces satisfying the $1\frac{1}{2}$-ball property we have that, in view of Corollary 2.5 and Proposition 2.8, every $|\cdot|$-proximinal subspace is complemented if and only if the absolute norm $|\cdot|$ is not of type 1.

If we apply consecutively the renorming processes in Corollary 2.6 and Proposition 2.8 we obtain the following result which includes both results as particular cases.

THEOREM 2.9. *Let $|\cdot|_1$ and $|\cdot|_2$ be type 1 absolute norms which are not of cotype ∞ and let M be a $|\cdot|_1$-proximinal subspace of X. Define $|||\cdot|||$ on X by*

$$|||x||| = \left|\left(\rho_M(x), \frac{n_1}{n_2} d(x, M)\right)\right|_2^+.$$

Then $|||\cdot|||$ is an equivalent norm on X and M is a $|\cdot|_2$-proximinal subspace in $(X, |||\cdot|||)$.

Recall [20, Sect. 1] that M is an absolute subspace of X if and only if M^0 is an absolute subspace of X^*. Similarly, M satisfies the $1\frac{1}{2}$-ball property in X if and only if M^0 satisfies it in X^* [24, Theorem 3]. We conclude our discussion of absolutely proximinal subspaces by showing that this class has the same desirable stability property. The first step in this direction is the following proposition which shows that the norm on a Banach space X which contains a $|\cdot|$-proximinal subspace M can be recovered from the seminorms ρ_M and $d(\cdot, M)$.

PROPOSITION 2.10. *Let M be a $|\cdot|$-proximinal subspace of X. Then*

$$\|x\| = |(\rho_M(x), d(x, M))|^+ \qquad \text{for all} \quad x \in X.$$

In particular, if M has the $1\frac{1}{2}$-ball property in X, then $\|x\| = \max\{\rho_M(x), d(x, M)\}$ for all $x \in X$.

Proof. Suppose that M is a $|\cdot|$-proximinal subspace of X and that $x \in X$.

If $\rho_M(x) \geq nd(x, M)$, then, using Theorem 2.3,

$$\begin{aligned}|(\rho_M(x), d(x, M))|^+ &= |(d(0, P_M(x)) + nd(x, M), d(x, M))|^+ \\ &= |(d(0, P_M(x)), d(x, M))| \\ &= \|x\|.\end{aligned}$$

If, on the other hand, $\rho_M(x) < nd(x, M)$, then Theorem 2.3 tells us that $0 \in P_M(x)$. Since $n^*(|\cdot|^+) \geq n(|\cdot|)$, we have $|(a, b)|^+ = b$ whenever $0 \leq a \leq nb$. In particular,

$$|(\rho_M(x), d(x, M))|^+ = d(x, M) = \|x\|,$$

as required. ∎

The proof of the next lemma requires another definition. We recall [10] that the duality mapping D is said to be (norm-to-norm) upper semicontinuous at some point u in the unit sphere of X if

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall y \in X$$

$$(\|y\| = 1 \text{ and } \|y - u\| < \delta) \Rightarrow D(y) \subset D(u) + B(0, \varepsilon).$$

LEMMA 2.11. *Let $u \in X$ with $\|u\| = 1$. If $\mathbf{K}u$ is an absolutely proximinal subspace of X, then the duality mapping is upper semicontinuous at u.*

Proof. Note that $N = \{x \in X : V(u, x) = \{0\}\}$ is a closed subspace of X. We let Y be the completion of the quotient space X/N with respect to the norm

$$\|x + N\| = \max\{|\lambda| : \lambda \in V(u, x)\}.$$

This norm will not generally agree with the usual quotient norm on X/N. It is easy to verify that $\|u + N\| = 1$ and that $\max\{|\lambda| : \lambda \in V(u + N, y)\} = \|y\|$ for all $y \in Y$. By [1, Corollary 5.9] the duality mapping on Y is norm-to-norm upper semicontinuous at $u + N$. Let us consider the Banach space $Y \times X/M$, where $M = \mathbf{K}u$, equipped with the norm given by

$$\|(y, x + M)\| = |(\|y\|, d(x, M))|^{+}.$$

(Here $|\cdot|$ is the absolute norm under which M is $|\cdot|$-proximinal.) A straightforward argument [10, Example 3.1] shows that the upper semicontinuity of the duality mapping on Y at $u + N$ implies the upper semicontinuity of the duality mapping on $Y \times X/M$ at $(u + N, 0)$.

Finally, by Proposition 2.10 the mapping $x \mapsto (x + N, x + M)$ is an isometric linear imbedding of X into $Y \times X/M$ which sends u to $(u + N, 0)$. The conclusion now follows from the fact that upper semicontinuity of duality mappings is preserved when we pass to subspaces. ∎

Observe that the full strength of absolute proximinality was not used in the previous proof, but only in the conclusion of Lemma 2.10. This property is studied in greater detail in Section 4.

THEOREM 2.12. *A subspace of a Banach space is an absolutely proximinal subspace if and only if its polar is absolutely proximinal in the dual space. More precisely, M is $|\cdot|$-proximinal in X if and only if M^0 is $|\cdot|^{*-}$-proximinal in X^*.*

Proof. ($\Rightarrow$) Let M be a $|\cdot|$-proximinal subspace of X. If the norm $|\cdot|$ is not of type 1, Corollary 2.5 tells us that M is a $|\cdot|$-summand. Then M^0 is a $|\cdot|^*$-summand and so by [20, Theorem 2.1] must be a $|\cdot|^*$ -proximinal subspace of X^*.

Now assume that $|\cdot|$ is of type 1. An application of Corollary 2.6 tells us that M satisfies the $1\frac{1}{2}$-ball property when X is renormed by

$$|||x||| = \max\{\rho_M(x), n\, \|x + M\|\}.$$

We have easily $|||m||| = \|m\|$ for all $m \in M$ and $|||x + M||| = n\, \|x + M\|$ for all $x \in X$. It is not difficult to check that if we now apply Proposition 2.8 to the

Banach space $(X, |||\cdot|||)$, in order to turn M into a $|\cdot|$-proximinal subspace, we get back the original norm $\|\cdot\|$. This means that $\|\cdot\|$ and $|||\cdot|||$ are also related by the identity,

$$\|x\| = \left|\left(|||x|||, \frac{1}{n}\, |||x + M||| \right)\right|^{-}.$$

We can now apply the dualization procedure used in the proof of [19, Theorem 2.3(b)], thereby obtaining for the dual norms of $\|\cdot\|$ and $|||\cdot|||$ the following relation:

$$\|f\| = \inf\{ |(n\, |||g|||,\, |||f + g|||)|^{+*} : g \in M^0 \} \qquad \forall f \in X^*.$$

This equality implies that the best approximant set for f in M^0 with respect to the norm $\|\cdot\|$ is the same as that for the norm $|||\cdot|||$, i.e., that P_{M^0} has an unambiguous meaning. Then we have clearly

$$\begin{aligned} \|f\| &\leqslant \inf\{ |(n\, |||g|||,\, |||f - g|||)|^{+*} : g \in P_{M^0}(f) \} \\ &= |(n d_1(0, P_{M^0}(f)),\, |||f + M^0|||)|^{+*}, \end{aligned}$$

where d_1 denotes the distance in the norm $|||\cdot|||$. From the identity $|||x + M||| = n\, \|x + M\|$ and the canonical identification of $(X/M)^*$ with M^0, we obtain $|||g||| = (1/n)\, \|g\|$ for all $g \in M^0$, so $n d_1(0, P_{M^0}(f)) = d(0, P_{M^0}(f))$. An analogous argument shows that $|||f + M^0||| = \|f + M^0\|$, in view of the fact that $|||m||| = \|m\|$ for all $m \in M$. Then the last inequality reads $\|f\| \leqslant |(d(0, P_{M^0}(f)),\, \|f + M^0\|)|^{+*}$. An elementary calculation with absolute norms shows that $|\cdot|^{+*} = |\cdot|^{*-}$. So we have

$$\|f\| \leqslant |(d(0, P_{M^0}(f)),\, \|f + M^0\|)|^{*-}.$$

We must prove that this inequality is in fact an equality. We clearly have

$$\begin{aligned} |(n\, |||g|||,\, |||f + g|||)|^{*-} &= |(n(|||g||| + |||f + g|||),\, |||f + g|||)|^{*} \\ &\geqslant |(n\, |||f|||,\, |||f + g|||)|^{*} \end{aligned}$$

for all $g \in M^0$. Taking the infimum over g, we obtain $\|f\| \geqslant |(n\, |||f|||,\, |||f + M^0|||)|^{*}$. Since M satisfies the $1\frac{1}{2}$-ball property in $(X, |||\cdot|||)$ we have that M^0 satisfies the same in $(X^*, |||\cdot|||)$ [24, Theorem 3]. Thus

$$\begin{aligned} \|f\| &\geqslant |(n d_1(0, P_{M^0}(f)) + n\, |||f + M^0|||,\, |||f + M^0|||)|^{*} \\ &= |(d(0, P_{M^0}(f)),\, \|f + M^0\|)|^{*-}, \end{aligned}$$

as required.

($\Leftarrow$) This part of the Theorem is more difficult and is broken into several steps.

First note that, by Proposition 1.5, we need to consider only the case $\dim(M^0) = 1$, for if M^0 is a $|\cdot|^{*-}$-proximinal subspace of X^* we apply the above result on quotients and find that M^0/Y^0 is a $|\cdot|^{*}$ -proximinal subspace of $X^*/Y^0 \cong Y^*$, for any closed subspace Y of X containing M. We apply this with $Y = M + \mathbf{K}x$ for arbitrary $x \in X$ and we are in the one-dimensional case which we suppose to be solved. So we obtain that M is a $|\cdot|$-proximinal subspace of $M + \mathbf{K}x$ for all $x \in X$, and this is just what we need.

If $|\cdot|^{*-} = |\cdot|^{+*}$ is not of type 1, then M^0 is a $|\cdot|^{-*}$-summand in X, and $|\cdot|^{+}$ is not of cotype ∞. From [19] it follows that M is a $|\cdot|^{+}$-summand in X, and by [20, Theorem 2.1] it must be $|\cdot|^{+}$ -proximinal. But $|\cdot|^{+} = |\cdot| = |\cdot|$, because $n^* + n = n^*(|\cdot|^{*}) = 0$, and thus M is $|\cdot|$-proximinal.

So we assume that $|\cdot|^{*}$ is of type 1 and that M^0 is one-dimensional. Thus $M^0 = \mathbf{K}g$ for some $g \in X^*$ with $\|g\| = 1$. An application of Corollary 2.6 shows that M^0 satisfies the $1\frac{1}{2}$-ball property in X^*, when the latter is normed by

$$|||f||| = \max\{\rho_{M^0}(f), n\, \|f + M^0\|\}.$$

This uses the fact that $n(|\cdot|^{*-}) = n(|\cdot|)$. Our next task is to establish that $|||\cdot|||$ is a dual norm on X^*. This is the deepest point in the proof.

Applying Lemma 2.11, together with [1, Theorems 3.4 and 5.1], we obtain

$$\begin{aligned} V(g, f) &= \{F(f) : F \in X^{**}, \|F\| = F(g) = 1\} \\ &= \overline{\{f(x) : x \in X, \|x\| = g(x) = 1\}} \qquad \forall f \in X^*. \end{aligned}$$

Now it is a matter of using straightforward calculations to verify that the closed unit ball for the norm $|||\cdot|||$ is w^*-closed. Then $|||\cdot|||$ is the dual norm of an equivalent norm on X which we denote also by $|||\cdot|||$. By [24, Theorem 3] M satisfies the $1\frac{1}{2}$-ball property in $(X, |||\cdot|||)$. Now we apply Proposition 2.8 to obtain yet another norm $\|\cdot\|_0$ on X, such that M is a $|\cdot|$-proximinal subspace of $(X, \|\cdot\|_0)$. The proof concludes by showing that $\|\cdot\| = \|\cdot\|_0$. To this end we use the defining formula for $\|\cdot\|_0$, that is,

$$\|x\|_0 = \left|\left(|||x|||, \frac{1}{n}\, |||x + M||| \right)\right|^{+}.$$

As in the proof of the "only if" part of the Theorem, we can dualize to obtain

$$\|f\|_0 = |(d(0, P_{M^0}(f)), \|f + M^0\|)|^{*} = \|f\|$$

for all $f \in X^*$. ∎

3. The $1\frac{1}{2}$-Ball Property in Complex Banach Spaces

How abundant are subspaces which are absolutely proximinal? In view of Proposition 2.7, this is essentially the same as asking what examples are known of subspaces with the $1\frac{1}{2}$-ball property. Of course all M-ideals and L-summands have this property, but we are interested in finding more general examples. Every subalgebra of $C_{\mathbf{R}}(K)$ (where K is a compact Hausdorff space) has the $1\frac{1}{2}$-ball property, but this is not true of those of $C_{\mathbf{C}}(K)$ [23, Proposition 2.5]. (However, self-adjoint subalgebras of $C_{\mathbf{C}}(K)$ are U-proximinal [9, Proposition 10].) Apart from the "easy" examples of M-ideals and L-summands, examples of subspaces of complex Banach spaces having the $1\frac{1}{2}$-ball property seem to be very rare.

Until recently only one example was known; $K(l_1)$ has the $1\frac{1}{2}$-ball property in $B(l_1)$, for either scalar field [23, Proposition 2.8]. Several other examples have now appeared [25], some of them closely related to this one.

In this section, we show that such examples are most abundant. In fact every complex Banach space has the $1\frac{1}{2}$-ball property in some superspace, in a nontrivial way. We present the results in a manner which is independent of the scalar field, since this result is also of some interest for real Banach spaces.

For sets A and B in some Banach space, we write $A \simeq B$ to mean that the two sets have the same closure and the same interior. Given $r > 0$, let us say that a set $S \subset \mathbf{K}$ is r-balanceable if there is another set $T \subset \mathbf{K}$ with $S + T \simeq \{\lambda \in \mathbf{K} : |\lambda| \leqslant r\}$. This property is not very interesting if $\mathbf{K} = \mathbf{R}$.

Lemma 3.1. *Let $M = \mathbf{K}u$ be a one-dimensional subspace of X with the $1\frac{1}{2}$-ball property. Assume $\|u\| = 1$, and write $P_M(x) = K(x)\,u$, where $K(x) \subset \mathbf{K}$. Then, for all $x \in X$, $K(x) - V(x)$ is the ball $B(0, d(x, M))$ in $\mathbf{K}$.*

Proof. We abbreviate $V(u, x)$ by writing $V(x)$. By Lemma 2.1 we have

$$M^u(x) = \inf\{M^u(y) : y \in P_M(x)\} + d(x, M)$$
$$= \inf\{\text{re}\,\lambda : \lambda \in K(x)\} + d(x, M),$$

that is,

$$\max\{\text{re}\,\mu : \mu \in V(x) - K(x)\} = d(x, M).$$

So the compact convex sets $V(x) - K(x)$ and $B_{\mathbf{K}}(0, d(x, M))$ have the same support mapping. ∎

Theorem 3.2. *Let Y be a real or complex Banach space and K a closed convex subset. Then the following are equivalent.*

(i) *For all $f \in Y^*$, $f(K)$ is $\|f\|$-balanceable.*

(ii) *There exist a Banach space X containing Y, an element $e \in X$ with $d(e, Y) = 1$ such that Y has the $1\frac{1}{2}$-ball property in X, and $P_Y(e) = K$.*

Proof. (i) ⇒ (ii) Given any $f \in Y^*$ with $\|f\| = 1$, we can find a set $S_f \subset \mathbf{K}$ with $f(K) - S_f \simeq B(0, 1)$. Let X be the vector space $Y \oplus \mathbf{K}e$, and define

$$\|y + \lambda e\| = \max\{|\lambda|, \sup_{\|f\| = 1} |f(y) + \lambda S_f|\}.$$

This is obviously a norm on X which coincides with the original norm on Y. Clearly $\|y - e\| \geqslant 1$ for all $y \in Y$, and

$$\begin{aligned} \|y - e\| = 1 &\Leftrightarrow f(y) - S_f \subseteq B_{\mathbf{K}}, &\quad \forall f \in B(0, 1), \\ &\Leftrightarrow f(y) \in \overline{f(K)}, &\quad \forall f \in Y^*, \\ &\Leftrightarrow y \in K. \end{aligned}$$

Thus $P_Y(e) = K$, whence $d(0, P(y + \lambda e)) = d(y, -\lambda K)$ and $d(y + \lambda e, Y) = |\lambda|$.

In order for us to establish L-proximinality, it is clearly sufficient to check that $\|y + \lambda e\| = d(0, P(y + \lambda e)) + d(y + \lambda e, Y)$, whenever $y \in Y$ and $\lambda = -1$. Thus we must establish the identity

$$\max\{1, \sup_{\|f\| = 1} |f(y) - S_f|\} = d(y, K) + 1.$$

This is clear if $y \in K$. Given $y \notin K$, we can certainly find an $f \in Y^*$ with $\|f\| = 1$ and $\inf \operatorname{re} f(K) - f(y) = d(y, K)$. But $\operatorname{re}(f(K) - S_f) \simeq (-1, 1)$, and so

$$\inf \operatorname{re} f(K) - \sup \operatorname{re} S_f = -1.$$

Thus

$$\sup \operatorname{re}(S_f - f(y)) = 1 + \inf \operatorname{re} f(K) - f(y) = d(y, K) + 1$$

and

$$\|y - e\| \geqslant |S_f - f(y)| \geqslant d(y, K) + 1.$$

The reverse inequality follows easily from the triangle inequality, so the proof is complete.

(ii) ⇒ (i) Fix $f \in Y^*$ with $\|f\| = 1$, and put $M = \ker f$. Then Y/M is a one-dimensional subspace of X/M, with the $1\frac{1}{2}$-ball property. Lemma 1.3 then tells us that

$$P_{Y/M}(e + M) = \overline{P_Y(e) + M} = \overline{f(K)},$$

where we have made use of the natural isomorphism between Y/M and the scalar field. The previous lemma now ensures that $f(K)$ is 1-balanceable. ∎

For real Banach spaces, a simpler proof of Theorem 3.2 is available. Note that hypothesis (i) is always satisfied. To establish (ii), we turn $X = Y \oplus \mathbf{R}e$ into a Banach space by defining $\|y - \lambda e\| = |\lambda| + d(y, \lambda K)$. It is easy to check that this defines a norm under which Y is L-proximinal in X. This argument does not work for complex scalars, since the term $d(y, \lambda K)$ might not be subadditive.

Recall that if Y is an M-ideal (respectively, an L-summand) in X and $x \in X \setminus Y$, then the linear span of $P_Y(x)$ is all of Y (respectively one-dimensional). The next result shows that there are abundant examples of absolutely proximinal subspaces with neither of the above properties.

Let us say that a subset S of a Banach space has constant width w if $S - S \simeq B(0, w)$. Clearly every ball has constant width, but there are asymmetric examples, the most famous of which is the Reuleaux triangle. This is the intersection, in the euclidean plane, of three balls of radius w, whose vertices form an equilateral triangle of side length w.

COROLLARY 3.3. *Let X be any Banach space, M any closed subspace whose dimension over the reals is at least two, and $|\cdot|$ an absolute norm of type 1 and not of cotype ∞. Then there is a Banach space Y containing X and a point $e \in Y$, such that X is $|\cdot|$-proximinal in Y and $P_X(e)$ is not symmetric and its linear span equals M.*

Proof. Choose subspaces E and F of M such that $M = E \oplus F$ and F has dimension one/two, depending on whether the scalars are complex/real. Let S be a Reuleaux triangle, of width $\frac{1}{2}$, in F. Then $K = \frac{1}{2}B_E + S$ certainly has the property that $f(K)$ is balanceable, for all $f \in X^*$. Theorem 3.2 establishes the result in the case $|\cdot| = L$, and the general statement then follows from Theorem 2.9. ∎

It is natural to ask which subsets of $\mathbf{C}$ are balanceable. Clearly every set of constant width is balanceable, and it might be conjectured that the converse is true. The following example shows this is not so.

EXAMPLE 3.4. Let a square be given in $\mathbf{C}$. Determine four points $x_1, ..., x_4$ inside the square and $r > 0$, such that for each i, two adjacent vertices of the square lie on the boundary of $B(x_i, r)$ and the arc joining them subtends an angle of $\pi/4$. Put $S = \bigcap_{i=1}^4 B(x_i, r)$, and let T be the body obtained from S by a rotation of $\pi/4$. It is easily checked that $S + T$ is a ball of radius r. Being symmetric, S does not have constant width.

4. Interior Points of $P(x)$

In this section, we define a new property of subspaces, much weaker than $|\cdot|$-proximinality, and use it to establish some estimates for the radii of balls contained in $P_M(x)$. This generalizes somewhat similar estimates obtained by Harmand [12] for the special case of M-ideals.

Given an absolute norm $|\cdot|$, let us say that M has the $|\cdot|$-property if, for all $x \in X$, $\|x\| = |(\rho_M(x), d(x, M))|$.

This property is quite weak. One can easily check that every Banach space has the M-property in its bidual, and so the M-property does not even imply proximinality. Nevertheless, it is a useful property for us to consider, as the remainder of this section shows.

Proposition 4.1. (i) *Given any absolute norm $|\cdot|$, every $|\cdot|$-proximinal subspace has the $|\cdot|^+$-property.*

(ii) *If $|\cdot|$ is an absolute norm not of type 1, then every $|\cdot|$-summand has the $|\cdot|$-property.*

(iii) *If $|\cdot|$ is a type 1 absolute norm, then no nontrivial subspace of any Banach space has the $|\cdot|$-property.*

Proof. (i) This is just a restatement of Proposition 2.10. We remark that $|\cdot|^+$ is never a type 1 norm, i.e., that $n(|\cdot|^+) = 0$ for any absolute norm $|\cdot|$.

(ii) Every $|\cdot|$-summand is $|\cdot|^-$-proximinal, by [20], and so has $|\cdot|^{-+}$-property. But $|\cdot|^{-+} = |\cdot|^+$, in general, and $|\cdot|^+ = |\cdot|$ when $n(|\cdot|) = 0$.

(iii) If $M \neq \{0\}$ has the $|\cdot|$-property in X, let us choose $u \in M$ with $\|u\| = 1$. For any $x \in X$ and $\alpha \in \mathbf{R}^+$, we have $\|u + \alpha x\| = |(\rho_M(u + \alpha x), \alpha d(x, M))|$. Computing right-hand derivatives at the origin, as in the proof of Lemma 2.1, we obtain the identity $M^u(x) = M^u(x) + nd(x, M)$. Thus $n = 0$, unless $M = X$. ∎

Let us remark that the proof of Lemma 2.11 used only the fact that M had the $|\cdot|^+$-property in X, not the full strength of $|\cdot|$-proximinality.

Corollary 4.2. *If a Banach space is absolutely proximinal in its bidual, then it has the $1\frac{1}{2}$-ball property in its bidual.*

Proof. If X is $|\cdot|$-proximinal in X^{**}, then $\|F\| = |(\rho_X(F), d(F, X))|^+$ for all $F \in X^{**}$. But $\rho_X(F) = \|F\|$ by the Hahn–Banach and Bishop–Phelps Theorems [19, Lemma 1.16]. So $|(a, b)|^+ = a$ whenever $0 \leqslant b \leqslant a$. This easily implies that $|\cdot|^+ = M$. Since $|\cdot|$ is not of cotype ∞, by Proposition 1.6, we must have $|\cdot| = L$. In other words, X has the $1\frac{1}{2}$-ball property in X^{**}. ∎

For Banach spaces which are absolute subspaces in their biduals, much more is known [6]. There are many Banach spaces which are M-ideals in their own biduals. For example, every subspace of the space of compact operators $K(l_p, l_q)$, for $1 < p \leqslant q < \infty$, has this property [13, Example 3.3(a) and Theorem 3.4]. It is well known that every $L_1(\mu)$ space is an L-summand in its own bidual, as is the predual of every von Neumann algebra [22, Theorem 3]. It follows from [23, Corollary 2.3] that every $C_{\mathbf{R}}(K)$ has the $1\frac{1}{2}$-ball property in its bidual. It would be interesting to have some more examples of Banach spaces which are absolutely proximinal in their biduals.

There is little point in defining an f-property in the manner of Section 1. If we define f on the positive quadrant by $f(a, b) = a$ for $0 \leqslant b \leqslant a$, and arbitrarily for $b > a \geqslant 0$, then every Banach space would have the f-property in its bidual. Thus there are no uniqueness results analogous to Lemma 1.2 and Proposition 1.6.

Given $M \subset X$, let us define two indices

$$\nu(X, M) = \max\{K : K\,\|x\| \leqslant \rho_M(x) \text{ for all } x \in X\}$$

and

$$\mu(X, M) = \sup\{\nu(Y, M) : M \subset Y \subseteq X\}.$$

Obviously $0 \leqslant \nu(X, M) \leqslant \mu(X, M) \leqslant 1$, and $\nu(X, M) = \mu(X, M)$ whenever M is a hyperplane in X. Also $\mu(X^{**}, X) = \nu(X^{**}, X) = 1$ for every Banach space X. On the other hand, equality is not usual. If A is a noncommutative, unital C^*-algebra, then $\nu(A, \mathbf{C}1) = \frac{1}{2}$ but $\mu(A, \mathbf{C}1) = 1$. This follows, for example, from [7, Theorem 3].

LEMMA 4.3. *Let M have the $|\cdot|$-property in X and $x \in X$. Then*

(i) *for $m \in M$, we have $m \in P_M(x) \Leftrightarrow \rho_M(x - m) \leqslant n^* d(x, M)$,*

(ii) $0 \notin P_M(x) \Rightarrow \rho_M(x) \geqslant n^*\,\|x\|$,

(iii) *if $\mu(X, M) < n^*$, then M is proximinal in X.*

Proof. (i) is clear from the identity

$$\|m - x\| = d(x, M)\left|\left(\frac{\rho_M(x - m)}{d(x, M)}, 1\right)\right|.$$

(ii) If $d(x, M) < \|x\|$ and $\rho_M(x) < n^*\,\|x\|$ we obtain the contradiction

$$\|x\| = |(\rho_M(x), d(x, M))| < |(n^*\,\|x\|, \|x\|)| = \|x\|.$$

(iii) Put $Y = M \oplus \mathbf{K}x$. (We assume that $x \notin M$, as otherwise $P_M(x)$ is obviously nonempty.) Since $\nu(Y, M) < n^*$, we can find $y \in Y$ with $\rho_M(y) <$

$n^* \|y\|$. Clearly $y \notin M$, so $x \in M \oplus \mathbf{K}y$. By (ii), $0 \in P_M(y)$, whence $P_M(x) \neq \varnothing$. ∎

We denote by $r_i(S)$ the inner radius of a set S, i.e., the supremum of those real numbers r for which S contains some ball of radius r. It is notationally convenient to adopt the convention that $r_i(\varnothing) = 0$.

THEOREM 4.4. *Suppose M has the $|\cdot|$-property in X. Then, for all $x \in X$,*

$$r_i(P_M(x)) \geqslant (n^* - \mu(X, M))\, d(x, M) \geqslant 0.$$

Proof. Given $M \subset Y \subseteq X$ and $x \in Y \backslash M$, we have

$$|x - m| = |(\rho_M(x - m), d(x, M))| \geqslant |(\nu(Y, M)\, \|x - m\|, d(x, M))|$$

for all $m \in M$. Taking the infimum yields

$$d(x, M) \geqslant |(\nu(Y, M)\, d(x, M), d(x, M))|,$$

and so $\nu(Y, M) \leqslant n^*$. The second inequality follows immediately.

The proof of the first inequality requires more delicacy. First we strengthen the inequality of Lemma 4.3(ii) to

$$\rho_M(x) \geqslant n^* \|x\| - r_i(P_M(x)).$$

This is clear if $0 \notin P_M(x)$, so assume that $0 \in P_M(x)$. Given $\varepsilon > 0$, choose $m \in M \backslash P_M(x)$ with $\|m\| < d(0, M \backslash P_M(x)) + \varepsilon$. Then, since $0 \notin P_M(x - m)$,

$$\begin{aligned} \rho_M(x) &\geqslant \rho_M(x - m) - \rho_M(m) \\ &\geqslant n^* \|x - m\| - \|m\| \\ &> n^* d(x, M) - d(0, M \backslash P_M(x)) - \varepsilon \\ &\geqslant n^* \|x\| - r_i(P_M(x)) - \varepsilon, \end{aligned}$$

as required.

Thus for any $m \in M$, $x \notin M$, $\alpha \in \mathbf{K}$,

$$\begin{aligned} \rho_M(\alpha x + m) &\geqslant n^* \|\alpha x + m\| - |\alpha|\, r_i(P_M(x)) \\ &\geqslant \left(n^* - \frac{r_i(P_M(x))}{d(x, M)}\right) \|\alpha x + m\|. \end{aligned}$$

Fixing $x \in X \backslash M$, we obtain

$$\mu(X, M) \geqslant \nu(M \oplus \mathbf{K}x, M) \geqslant n^* - \frac{r_i(P_M(x))}{d(x, M)},$$

thereby establishing the first inequality. (The case $x \in M$ is clear.) ∎

THEOREM 4.5. *Let M have the $|\cdot|$-property in X and $x \in X$. Then*

$$r_i(P_M(x)) \leqslant (n^* - v(X, M))\, d(x, M).$$

Proof. Suppose that $B(a, r) \subseteq P_M(x)$. Then, for any $f \in D(M) = \bigcup_{|u|=1} D(M, u)$,

$$m \in M, \qquad \|m\| < r \Rightarrow a - m \in P_M(x)$$
$$\Rightarrow \rho_M(a - x - m) \leqslant n^* d(x, M)$$
$$\Rightarrow |f(a-x) - f(m)| \leqslant n^* d(x, M).$$

Hence, for any $f \in D(M)$, $|f(a-x)| + r \leqslant n^* d(x, M)$, i.e., $\rho_M(a-x) + r \leqslant n^* d(x, M)$. But $\rho_M(a-x) \geqslant v(X, M)\, |a-x| = v(X, M)\, d(x, M)$, so $r \leqslant (n^* - v(X, M))\, d(x, M)$. ∎

COROLLARY 4.6. *Let M be a subspace of X with $\mu(X, M) = v(X, M)$. (In particular, suppose that M is a hyperplane in X.)*

(i) *If M has the $|\cdot|$-property in X, then $r_i(P_M(x)) = (n^* - v(X, M))\, d(x, M)$ for all $x \in X$.*

(ii) *If M is $|\cdot|$-proximinal in X, then $r_i(P_M(x)) = (n - v(X, M))\, d(x, M)$ for all $x \in X$.*

The hypotheses of Corollary 4.6 also apply when $X = M^{**}$, but this situation is not very interesting. For the special case of M-ideals, Corollary 4.6 was first proved by Harmand ([12, Kapitel II] or [3, Sect. 5]). He defined an index for M-ideals, called the grade, which is equal to our $v(X, M)$ in this case. He showed that $v(X, M) = 1 - r_i(P_M(x))$ whenever M is an M-ideal of codimension one in X, and $d(x, M) = 1$. (Since every M-ideal has the M-property, $n^* = 1$.)

Holmes *et al.* [14, Sect. 4] noted that every M-summand M of X satisfies int $M^{-} \neq \varnothing$, and that in certain classical examples of M-ideals, we had int $M^{-} = \varnothing$. This led them to ask whether every proper M-ideal has the property that its metric complement has an empty interior. There are several ways to see that this is not so.

For a counterexample in a classical Banach space, let K be a compact, Hausdorff space, and K_0 a closed subset of K. Then $M = \{f \in C(K): f(K_0) = 0\}$ is easily checked to be an M-ideal in $C(K)$, which is proper whenever K_0 is not clopen. It follows from Urysohn's Lemma that M^{-} has nonempty interior (in $C(K)$) if and only if K_0 has nonempty interior (in K). We are indebted to D. Werner for bringing this result to our attention.

More generally, it is observed in [3, Sect. 5] that the grade of an M-ideal can always be decreased. More precisely, this means that if M is an M-ideal in X and $\alpha < v(X, M)$, then there is a Banach space X_α containing M as an

M-ideal, with $\nu(X_\alpha, M) = \alpha$. This shows that $r_i(P_M(x))$ can be increased, which is tantamount to adding interior points to M^-, while preserving the proper M-idealness. In fact, it follows from [2, Proposition 2.2] that $\nu(X, M) = 0$ if and only if M is an M-summand in X. Thus the properness of M-ideals is characterized by "$r_i(P_M(x)) < d(x, M)$ for some $x \in X$," not by "$r_i(P_M(x)) = 0$, for all $x \notin M$."

We use these ideas to make some remarks about proper semi-M-ideals. It is now high time for us to define semi-ideals and absolute subspaces. Recall that M is said to be a semi-$|\cdot|$-summand of X if there is a mapping π from X onto M satisfying the identities

$$\pi(\lambda x + \pi(y)) = \lambda\pi(x) + \pi(y)$$

and

$$\|x\| = |(\|\pi(x)\|, \|x - \pi(x)\|)|.$$

A semi-$|\cdot|$-ideal is a closed subspace M of X such that M^0 is a semi-$|\cdot|^*$-summand of X^*. Finally M is said to be a $|\cdot|$-subspace of X (or simply an absolute subspace if $|\cdot|$ need not be emphasized) when it is a semi-$|\cdot|$-ideal of $M + \mathbf{K}x$ for all $x \in X$. Semi-$|\cdot|$-ideals (hence $|\cdot|$-ideals and $|\cdot|$-summands) and semi-$|\cdot|$-summands (even semi-$|\cdot|$-idealoids [19]) are always $|\cdot|$-subspaces. On the other hand, every $|\cdot|$-subspace is a $|\cdot|^-$-proximinal subspace [20, Theorem 2.1] and hence has the $|\cdot|^+$-property. (Note that $|\cdot|^{-+} = |\cdot|^+$ for any absolute norm.) A semi-M-ideal is called proper if it is not an M-summand.

PROPOSITION 4.7. *If M is a proper semi-M-ideal of codimension one in X, then, for all $x \in X$,*

$$r_i(P_M(x)) = \tfrac{1}{2}(1 - \nu(X, M)) \operatorname{diam} P_M(x).$$

If in addition X is n-dimensional $(n < \infty)$ then $0 < \nu(X, M) \leqslant 1 - 2/n$, and these estimates are the best possible.

Proof. It is well known [17, Theorem 1.2] that $P_M(x) - P_M(x) \simeq B(0, 2d(x, M))$. The first statement then follows from Corollary 4.6. It is a well-known consequence of Helly's Theorem that, in any n-dimensional normed space, any set of constant width w contains a ball of radius $w/(n+1)$. Thus $r_i(P_M(x)) \geqslant 2d(x, M)/n$. Propriety forces $r_i(P_M(x)) < 1$.

The renorming process of Harmand shows that $r_i(P_M(x))$ can be arbitrarily close to 1. (Of course equality holds precisely when M is an M-summand in X.) In the other direction, the classical example $X = l_1(n)$ and $M = \{(x_1, ..., x_n) : x_1 + \cdots + x_n = 0\}$ shows that $r_i(P_M(x)) = 2/n$ is possible, when $x = (1/n)(1, 1, ..., 1)$. ∎

We finish with a couple of results about absolute subspaces. For further information about absolute subspaces we refer to [6] and [20].

PROPOSITION 4.8. *For a fixed absolute norm* $|\cdot|$, *M is a semi-*$|\cdot|$*-summand of X if and only if M is a* $|\cdot|$*-subspace of X with* $\mu(X, M) = n$.

Proof. Necessity is clear from [20, Theorem 1.7]. For sufficiency, fix $x \in X$. Since M has the $|\cdot|^+$-property,

$$\begin{aligned} r_i(P_M(x)) &\geqslant (n^*(|\cdot|^+) - \mu(X, M))\, d(x, M) \\ &= (n^*(|\cdot|^+) - n)\, d(x, M) \\ &= n^* d(x, M) \\ &= \tfrac{1}{2}\operatorname{diam}(P_M(x)). \end{aligned}$$

The last equality follows from [20, Theorem 2.1]. It is clear that $P_M(x)$ is a ball, so an application of [20, Corollary 2.2] shows that M is a semi-$|\cdot|$-summand. ∎

Our last result is an easy consequence of Corollary 4.6(i) and the arguments used above.

COROLLARY 4.9. *If M is an* $|\cdot|$*-subspace of X, with* $\mu(X, M) = \nu(X, M)$, *then for all* $x \in X$, *we have* $2n^* r_i(P_M(x)) = (n^* + n - \nu(X, M)) \operatorname{diam} P_M(x)$.

In this regard, it is pretty obvious that if M is a proximinal subspace of codimension one in X, then $\operatorname{diam} P_M(x)/d(x, M)$ and $r_i(P_M(x))/d(x, M)$ are both independent of $x \in X \backslash M$. If M is a $|\cdot|$-subspace, of any codimension, then $\operatorname{diam} P_M(x)/d(x, M) = 2n^*$ for all $x \notin M$ [20, Theorem 2.1], but $r_i(P_M(x))/d(x, M)$ may vary with x. Finally, the example $\mathbf{R}(1, 1, 0) \subset l_\infty(3)$ shows that the $1\frac{1}{2}$-ball property is not sufficient to guarantee that $\operatorname{diam} P_M(x)/d(x, M)$ is constant.

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